Segre variety, conifold, Hopf fibration, and separable multi-qubit states

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Abstract

We establish relations between Segre variety, conifold, Hopf fibration, and separable sets of pure two-qubit states. Moreover, we investigate the geometry and topology of separable sets of pure multi-qubit states based on a complex multi-projective Segre variety and higher order Hopf fibration.

1 Introduction

Quantum entanglement[1, 2] is one of the most interesting features of quantum theory. In quantum mechanics, the space of pure states in is an $N+1$-dimensional Hilbert space can be described by the complex projective space $\mathbf{CP}^N$. For bipartite, pure states, the entanglement of formation can be written in terms of concurrence [3]. The connection between concurrence and geometry is found in a map called Segre embedding, see D. C. Brody and L. P. Hughston [4]. They illustrate this map for a pair of qubits, and point. The Segre embedding has also been discussed in [5]. There is also another geometrical description to describe pure state called Hopf fibration. The relation between Hopf fibration and single qubit and two-qubit states is discussed by R. Mosseri and R. Dandoloff [6]. They have shown that $\mathbf{S}^2$ base space of a suitably oriented $\mathbf{S}^3$ Hopf fibration is nothing but the Bloch sphere, while the circular fibres represent the qubit overall phase degree of freedom. For two-qubit states, the Hilbert space is a seven-dimensional sphere $\mathbf{S}^7$, which also allows for a second Hopf fibration which is entanglement sensitive, with $\mathbf{S}^3$ fibres and a $\mathbf{S}^4$ base. Moreover, a generalization of Hopf fibration to three-qubit state has been presented in Ref. [7], where the Hilbert space of the three-qubit state is the fifteen-dimensional sphere $\mathbf{S}^{15}$, which allows for the third Hopf fibration with $\mathbf{S}^8$ as base and $\mathbf{S}^7$ as fiber. In this paper we will describe the Segre variety, which is a quadric space in algebraic geometry [8, 9, 10, 11, 12], by giving a complete and explicit formula for it. We will compare the Segre variety with the concurrence of pure, two-qubit states. The vanishing of the concurrence of a pure two-qubit state coincides with the Segre variety. Moreover, we will establish relations between Segre variety, conifold and Hopf fibration. In algebraic geometry, a conifold is a generalization of the notion of a manifold. Unlike manifolds, a conifold can contain conical singularities, i.e., points whose neighborhood look like a cone with a certain base. The base is usually a five-dimensional manifold. Conifold are very important in string theory, i.e., in the process of compactification of Calabi-Yau

manifolds. A Calabi-Yau manifold is a compact Kähler manifold with a vanishing first Chern class. A Calabi-Yau manifold can also be defined as a compact Ricci-flat Kähler manifold. Finally, we will discuss the geometry and topology of pure multi-qubit states based on some mathematical tools from algebraic geometry and algebraic topology, namely the multi-projective Segre variety and higher-order Hopf fibration. Let us start by denoting a general, pure, composite quantum system with m subsystems $\mathcal{Q} = \mathcal{Q}^p_m(N_1, N_2, \ldots, N_m) = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$, consisting of a pure state $|\Psi\rangle = \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}\cdots\sum_{i_m=1}^{N_m}\alpha_{i_1,i_2,\ldots,i_m}|i_1, i_2, \ldots, i_m\rangle$ and corresponding Hilbert space as $\mathcal{H}_\mathcal{Q} = \mathcal{H}_{\mathcal{Q}_1}\otimes\mathcal{H}_{\mathcal{Q}_2}\otimes\cdots\otimes\mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the jth Hilbert space is given by $N_j = \dim(\mathcal{H}_{\mathcal{Q}_j})$. We are going to use this notation throughout this paper, i.e., we denote a pure two-qubit states by $\mathcal{Q}^p_2(2,2)$. Next, let $\rho_\mathcal{Q}$ denotes a density operator acting on $\mathcal{H}_\mathcal{Q}$. The density operator $\rho_\mathcal{Q}$ is said to be fully separable, which we will denote by $\rho^{sep}_\mathcal{Q}$, with respect to the Hilbert space decomposition, if it can be written as $\rho^{sep}_\mathcal{Q} = \sum_{k=1}^{\mathrm{N}} p_k \bigotimes_{j=1}^m \rho^k_{\mathcal{Q}_j}$, $\sum_{k=1}^N p_k = 1$ for some positive integer N, where p_k are positive real numbers and $\rho^k_{\mathcal{Q}_j}$ denotes a density operator on Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$. If $\rho^p_\mathcal{Q}$ represents a pure state, then the quantum system is fully separable if $\rho^p_\mathcal{Q}$ can be written as $\rho^{sep}_\mathcal{Q} = \bigotimes_{j=1}^m \rho_{\mathcal{Q}_j}$, where $\rho_{\mathcal{Q}_j}$ is a density operator on $\mathcal{H}_{\mathcal{Q}_j}$. If a state is not separable, then it is said to be entangled state.

2 Complex projective variety

In this section we will review basic definition of complex projective variety. Let $\{f_1, f_2, \ldots, f_q\}$ be continuous functions $\mathbf{K}^n \longrightarrow \mathbf{K}$, where $\mathbf{K}$ is field of real $\mathbf{R}$ or complex number $\mathbf{C}$. Then we define real (complex) space as the set of simultaneous zeroes of the functions

$$\mathcal{V}_\mathbf{K}(f_1, f_2, \ldots, f_q) = \{(z_1, z_2, \ldots, z_n) \in \mathbf{K}^n : f_i(z_1, z_2, \ldots, z_n) = 0 \ \forall \ 1 \leq i \leq q\}. \tag{1}$$

These real (complex) spaces become a topological spaces by giving them the induced topology from $\mathbf{K}^n$. Now, if all f_i are polynomial functions in coordinate functions, then the real (complex) space is called a real (complex) affine variety. A complex projective space $\mathbf{CP}^n$ which is defined to be the set of lines through the origin in $\mathbf{C}^{n+1}$, that is, $\mathbf{CP}^n = (\mathbf{C}^{n+1} - 0)/\sim$, where $\sim$ is an equivalence relation define by $(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1}) \Leftrightarrow \exists\lambda \in \mathbf{C} - 0$ such that $\lambda x_i = y_i \forall \ 0 \leq i \leq n$. For $n = 1$ we have a one dimensional complex manifold $\mathbf{CP}^1$ which is very important one, since as a real manifold it is homeomorphic to the 2-sphere $\mathbf{S}^2$. Moreover every complex compact manifold can be embedded in some $\mathbf{CP}^n$. In particular, we can embed a product of two projective spaces into a third one. Let $\{f_1, f_2, \ldots, f_q\}$ be a set of homogeneous polynomials in the coordinates $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ of $\mathbf{C}^{n+1}$. Then the projective variety is defined to be the subset

$$\mathcal{V}(f_1, f_2, \ldots, f_q) = \{[\alpha_1, \ldots, \alpha_{n+1}] \in \mathbf{CP}^n : f_i(\alpha_1, \ldots, \alpha_{n+1}) = 0 \ \forall \ 1 \leq i \leq q\}. \tag{2}$$

We can view the complex affine variety $\mathcal{V}_\mathbf{C}(f_1, f_2, \ldots, f_q) \subset \mathbf{C}^{n+1}$ as complex cone over projective variety $\mathcal{V}(f_1, f_2, \ldots, f_q)$. We can also view $\mathbf{CP}^n$ as a quotient of the unit $2n+1$ sphere in $\mathbf{C}^{n+1}$ under the action of $U(1) = \mathbf{S}^1$, that is $\mathbf{CP}^n = \mathbf{S}^{2n+1}/U(1) = \mathbf{S}^{2n+1}/\mathbf{S}^1$, since every line in $\mathbf{C}^{n+1}$ intersects the unit sphere in a circle.

3 Hopf fibration and two- and three-qubit states

For a pure one-qubit state $\mathcal{Q}_1^p(2)$ with $|\Psi\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle$, where $\alpha_1, \alpha_2 \in \mathbf{C}$, and $|\alpha_1|^2 + |\alpha_2|^2 = 1$, we can parameterize this state as

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \cos(\frac{\vartheta}{2}) \exp\left(i(\frac{\varphi}{2} + \frac{\chi}{2})\right) \\ \cos(\frac{\vartheta}{2}) \exp\left(i(\frac{\varphi}{2} - \frac{\chi}{2})\right) \end{pmatrix} \tag{3}$$

where $\vartheta \in [0, \pi]$, $\varphi \in [0, 2\pi]$ and $\chi \in [0, 2\pi]$. The Hilbert space $\mathcal{H}_\mathcal{Q}$ of a single qubit is the unit 3-dimensional sphere $\mathbf{S}^3 \subset \mathbf{R}^4 = \mathbf{C}^2$. But since quantum mechanics is $U(1)$ projective, the projective Hilbert space is defined up to a phase $\exp(i\varphi)$, so we have $\mathbf{CP}^1 = \mathbf{S}^3/U(1) = \mathbf{S}^3/\mathbf{S}^1 = \mathbf{S}^2$. Now, the first Hopf map, $\mathbf{S}^3 \xrightarrow{\mathbf{S}^1} \mathbf{S}^2$ as an $\mathbf{S}^1$ fibration over a base space $\mathbf{S}^2$. For a pure two-qubit state $\mathcal{Q}_2^p(2,2)$ with $|\Psi\rangle = \alpha_{1,1}|1,1\rangle + \alpha_{1,2}|1,2\rangle + \alpha_{2,1}|2,1\rangle + \alpha_{2,2}|2,2\rangle$, where $\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2} \in \mathbf{C}$ and $\sum_{k,l}^2 |\alpha_{k,l}|^2 = 1$. The normalization condition identifies the Hilbert space $\mathcal{H}_\mathcal{Q}$ to be the seven dimensional sphere $\mathbf{S}^7 \subset \mathbf{R}^8 = \mathbf{C}^4$ and the projective Hilbert space to be $\mathbf{CP}^3 = \mathbf{S}^7/\mathrm{U}(1)$. Thus we can parameterized the sphere $\mathbf{S}^7$ as a $\mathbf{S}^3$ fiber over $\mathbf{S}^4$, that is $\mathbf{S}^7 \xrightarrow{\mathbf{S}^3} \mathbf{S}^4$ which is called the Hopf second fibration. This Hopf map is entanglement sensitive and the separable states satisfy $\alpha_{1,1}\alpha_{2,2} = \alpha_{1,2}\alpha_{2,1}$, see Ref. [6].

4 Segre variety for a general bipartite state and concurrence

For given general pure bipartite state $\mathcal{Q}_2^p(N_1, N_2)$ we want make $\mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1}$ into a projective variety by its Segre embedding which we construct as follows. Let $(\alpha_1, \alpha_2, \ldots, \alpha_{N_1})$ and $(\alpha_1, \alpha_2, \ldots, \alpha_{N_2})$ be two points defined on $\mathbf{CP}^{N_1-1}$ and $\mathbf{CP}^{N_2-1}$, respectively, then the Segre map

$$\mathcal{S}_{N_1,N_2} : \mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \longrightarrow \mathbf{CP}^{N_1N_2-1} \tag{4}$$

$$((\alpha_1, \ldots, \alpha_{N_1}), (\alpha_1, \ldots, \alpha_{N_2})) \longmapsto (\alpha_{1,1}, \ldots, \alpha_{1,N_1}, \ldots, \alpha_{N_1,1}, \ldots, \alpha_{N_1,N_2})$$

is well defined. Next, let $\alpha_{i,j}$ be the homogeneous coordinate function on $\mathbf{CP}^{N_1N_2-1}$. Then the image of the Segre embedding is an intersection of a family of quadric hypersurfaces in $\mathbf{CP}^{N_1N_2-1}$, that is

$$\mathrm{Im}(\mathcal{S}_{N_1,N_2}) = \ < \alpha_{i,k}\alpha_{j,l} - \alpha_{i,l}\alpha_{j,k} >= \mathcal{V}\left(\alpha_{i,k}\alpha_{j,l} - \alpha_{i,l}\alpha_{j,k}\right). \tag{5}$$

This quadric space is the space of separable states and it coincides with the definition of general concurrence $\mathcal{C}(\mathcal{Q}_2^p(N_1, N_2))$ of a pure bipartite state [13, 14] because

$$\mathcal{C}(\mathcal{Q}_2^p(N_1, N_2)) = \left(\mathcal{N} \sum_{j,i=1}^{N_1} \sum_{l,k=1}^{N_2} \left|\alpha_{i,k}\alpha_{j,l} - \alpha_{i,l}\alpha_{j,k}\right|^2\right)^{\frac{1}{2}}, \tag{6}$$

where $\mathcal{N}$ is a somewhat arbitrary normalization constant. The separable set is defined by $\alpha_{i,k}\alpha_{j,l} = \alpha_{i,l}\alpha_{j,k}$ for all i, j and k, l. I.e., for a two qubit state we have $\mathcal{S}_{2,2} : \mathbf{CP}^1 \times \mathbf{CP}^1 \longrightarrow \mathbf{CP}^3$ and

$$\mathrm{Im}(\mathcal{S}_{2,2}) = \mathcal{V}\left(\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1}\right) \Longleftrightarrow \alpha_{1,1}\alpha_{2,2} = \alpha_{1,2}\alpha_{2,1} \tag{7}$$

is a quadric surface in $\mathbf{CP}^3$ which coincides with the space of separable set of pairs of qubits. In following section comeback to this result.

5 Conifold

In this section we will give a short review of conifold. An example of real (complex) affine variety is conifold which is defined by

$$\mathcal{V}_{\mathbf{C}}(\sum_{i=1}^{4} z_i^2) = \{(z_1, z_2, z_3, z_4) \in \mathbf{C}^4 : \sum_{i=1}^{4} z_i^2 = 0\}. \tag{8}$$

and conifold as a real affine variety is define by

$$\mathcal{V}_{\mathbf{R}}(f_1, f_2) = \{(x_1, \ldots, x_4, y_1, \ldots, y_4) \in \mathbf{R}^8 : \sum_{i=1}^{4} x_i^2 = \sum_{j=1}^{4} y_j^2, \sum_{i=1}^{4} x_i y_i = 0\}. \tag{9}$$

where $f_1 = \sum_{i=1}^{4}(x_i^2 - y_i^2)$ and $f_2 = \sum_{i=1}^{4} x_i y_i$. This can be seen by defining $z = x + iy$ and identifying imaginary and real part of equation $\sum_{i=1}^{4} z_i^2 = 0$. As a real topological space $\mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n) \subset \mathbf{R}^n$, $x \in \mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n)$ is a smooth point of $\mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n)$ if there is a neighborhood V of x such that V is homeomorphic to $\mathbf{R}^d$ for some d which is usually called the local dimension of $\mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n)$ in x. If there is no such neighborhood V, then x is said to be a singular point of $\mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n)$. Now, we can call $\mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n)$ a topological manifold if all points $x \in \mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n)$ are smooth. $\mathbf{S}^n$ is compact, since it is a closed and bounded subset of $\mathbf{R}^{n+1}$. Now, let us define a cone as a real space $\mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n) \subset \mathbf{R}^n$ with a specified point s such that for all $x \in \mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n)$ we have that the line $sx \in \mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n)$. But every line $s \in \mathbf{R}^n$ intersect any sphere $\mathbf{S}^{n-1}$ with center s, the cone $\mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n)$ can be determined by a compact space $\mathcal{B} = \mathcal{V}_{\mathbf{R}}(f_1, \ldots, f_n) \cap \mathbf{S}^{n-1}$ called the base space of the cone. As a real space, the conifold is cone in $\mathbf{R}^8$ with top the origin and base space the compact manifold $\mathbf{S}^2 \times \mathbf{S}^3$. One can reformulate this relation in term of a theorem. The conifold $\mathcal{V}_{\mathbf{C}}(\sum_{i=1}^{4} z_i^2)$ is the complex cone over the Segre variety $\mathbf{CP}^1 \times \mathbf{CP}^1 \simeq \mathbf{S}^2 \times \mathbf{S}^2$. To see this let us make a complex linear change of coordinate $\alpha'_{1,1} = z_1 + iz_2$, $\alpha'_{1,2} = -z_4 + iz_3$, $\alpha'_{2,1} = z_4 + iz_3$, and $\alpha'_{2,2} = z_1 - iz_2$. Thus after this linear coordinate transformation we have

$$\mathcal{V}_{\mathbf{C}}(\alpha'_{1,1}\alpha'_{2,2} - \alpha'_{1,2}\alpha'_{2,1}) = \mathcal{V}_{\mathbf{C}}(\sum_{i=1}^{4} z_i^2) \subset \mathbf{C}^4. \tag{10}$$

We will comeback to this result in section 6 where we establish a relation between these varieties, Hopf fibration and two-qubit state. Moreover, removal of singularity of a conifold leads to a Segre variety which also describes the separable two-qubit states. We will investigate this connection in the following

section. We can also define a metric on conifold as $dS_6^2 = dr^2 + r^2 dS^2_{T^{1,1}}$, where

$$dS^2_{T^{1,1}} = \frac{1}{9}\left(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i\right)^2 + \frac{1}{6}\sum_{i=1}^{2}\left(d\phi_i^2 + \sin^2\theta_i d\phi_i^2\right)^2, \qquad (11)$$

is the metric on the Einstein manifold $T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)}$, with $U(1)$ being a diagonal subgroup of the maximal torus of $SU(2)\times SU(2)$. Moreover, $T^{1,1}$ is a $U(1)$ bundle over $\mathbf{S}^2\times\mathbf{S}^2$, where $0 \leq \psi \leq 4$ is an angular coordinate and (θ_i, ϕ_i) for all $i = 1, 2$ parameterize the two $\mathbf{S}^2$, see Ref. [15, 16]. One can even relate these angular coordinate to the $\alpha^{'}_{k,l}$ for all $k, l = 1, 2$ as follows

$$\begin{array}{ll} \alpha^{'}_{1,1} = r^{3/2} e^{\frac{i}{2}(\psi-\phi_1-\phi_2)} \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} & \alpha^{'}_{1,2} = r^{3/2} e^{\frac{i}{2}(\psi+\phi_1-\phi_2)} \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \\ \alpha^{'}_{2,1} = r^{3/2} e^{\frac{i}{2}(\psi-\phi_1+\phi_2)} \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} & \alpha^{'}_{2,2} = r^{3/2} e^{\frac{i}{2}(\psi+\phi_1+\phi_2)} \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \end{array}.$$

Moreover, if we define the conifold as $\mathcal{V}_{\mathbf{C}}(\sum_{i=1}^{4} z_i^2)$, then we identify the Einstein manifold $T^{1,1}$ as the intersection of conifold with the variety $\mathcal{V}_{\mathbf{C}}(\sum_{i=1}^{4}|z_i^2| - r^3)$ and $T^{1,1}$ is invariant under rotations $SO(4) = SU(2)\times SU(2)$ of z_i coordinate and under an overall phase rotation.

6 Conifold, Segre variety, and a pure two-qubit state

In this section we will investigate relations between pure two-qubit states, Segre variety, and conifold. For a pure two-qubit state the Segre variety is given by $\mathcal{S}_{2,2} : \mathbf{CP}^1 \times \mathbf{CP}^1 \longrightarrow \mathbf{CP}^3$ and

$$\begin{aligned} \mathrm{Im}(\mathcal{S}_{2,3}) &= \mathcal{V}\left(\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1}\right) &(12)\\ &= \mathcal{V}(\alpha^{'2}_{1,1} + \alpha^{'2}_{2,2} + \alpha^{'2}_{1,2} + \alpha^{'2}_{2,1}) \\ &= \mathbf{CP}^1 \times \mathbf{CP}^1 \simeq \mathbf{S}^2 \times \mathbf{S}^2 \\ &\subset \mathbf{S}^4 \xleftarrow{\mathbf{S}^3} \mathbf{S}^7 \xrightarrow{\mathbf{S}^1} \mathbf{S}^7/U(1) = \mathbf{CP}^3 . \end{aligned}$$

where we have performed a coordinate transformation on ideal of Segre variety $\mathrm{Im}(\mathcal{S}_{2,2})$. Moreover, we have the following commutative diagram

$$\begin{tikzcd} \mathbf{S}^7 \arrow[r, "id"] \arrow[d, "\mathbf{S}^1"'] & \mathbf{S}^7 \arrow[d, "\mathbf{S}^3"] \\ \mathbf{CP}^3 = \mathbf{S}^7/U(1) \arrow[r, "\mathbf{S}^2"] & \mathbf{S}^4 = \mathbf{S}^7/SU(2) = \mathbf{HP}^1 \end{tikzcd}$$

where $\mathbf{HP}^1$ denotes projective space over quaternion number field and we have the second Hopf fibration $\mathbf{S}^7 \xrightarrow{\mathbf{S}^3} \mathbf{S}^4$. Thus we have established a direct relation between two-qubit state, Segre variety, conic variety and Hopf fibration. Thus the result from algebraic geometry and algebraic topology give a unified picture of two-qubit state. Now, let us investigate what happens to our state, when we do the coordinate transformation to establish relation between conic variety and Segre variety. By the coordinate transformation $\alpha^{'}_{1,1} = \alpha_{1,1} + i\alpha_{1,2}$,

$\alpha^{'}_{1,2} = -\alpha_{2,2} + i\alpha_{2,1}$, $\alpha^{'}_{2,1} = \alpha_{2,2} + i\alpha_{2,1}$, and $\alpha^{'}_{2,2} = \alpha_{1,1} - i\alpha_{1,2}$ we perform the following map $|\Psi\rangle = \alpha_{1,1}|1,1\rangle + \alpha_{1,2}|1,2\rangle + \alpha_{2,1}|2,1\rangle + \alpha_{2,2}|2,2\rangle \longrightarrow |\Psi^{'}\rangle$ which is given by

$$\begin{aligned}
|\Psi^{'}\rangle &= \alpha^{'}_{1,1}|1,1\rangle + \alpha^{'}_{1,2}|1,2\rangle + \alpha^{'}_{2,1}|2,1\rangle + \alpha^{'}_{2,2}|2,2\rangle \\
&= \alpha_{1,1}(|1,1\rangle + |2,2\rangle) + i\alpha_{1,2}(|1,1\rangle - |2,2\rangle) \\
&\quad + i\alpha_{2,1}(|1,2\rangle + |2,1\rangle) - \alpha_{2,2}(|1,2\rangle - |2,1\rangle) \\
&= \sqrt{2}\left(\alpha_{1,1}|\Psi^{+}\rangle + i\alpha_{1,2}|\Psi^{-}\rangle + i\alpha_{2,1}|\Phi^{+}\rangle - \alpha_{2,2}|\Phi^{-}\rangle\right).
\end{aligned} \tag{13}$$

Thus the equality between Segre variety, conic variety means that we rewrite a pure two-qubit state in terms of Bell's basis. For higher dimensional space we have Segre variety but we couldn't find any relation between these two variety.

7 Segre variety, Hopf fibration, and multi-qubit states

In this section, we will generalize the Segre variety to a multi-projective space and then we will establish connections between Segre variety for multi-qubit state and Hopf fibration. As in the previous section, we can make $\mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \times \cdots \times \mathbf{CP}^{N_m-1}$ into a projective variety by its Segre embedding following almost the same procedure. Let $(\alpha_1, \alpha_2, \ldots, \alpha_{N_j})$ be points defined on $\mathbf{CP}^{N_j-1}$. Then the Segre map

$$\begin{array}{rcl}
\mathcal{S}_{N_1,\ldots,N_m} : \mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \times \cdots \times \mathbf{CP}^{N_m-1} & \longrightarrow & \mathbf{CP}^{N_1 N_2 \cdots N_m - 1} \\
((\alpha_1, \alpha_2, \ldots, \alpha_{N_1}), \ldots, (\alpha_1, \alpha_2, \ldots, \alpha_{N_m})) & \longmapsto & (\ldots, \alpha_{i_1,i_2,\ldots,i_m}, \ldots).
\end{array} \tag{14}$$

is well defined for $\alpha_{i_1,i_2,\ldots,i_m}$, $1 \leq i_1 \leq N_1, 1 \leq i_2 \leq N_2, \ldots, 1 \leq i_m \leq N_m$ as a homogeneous coordinate-function on $\mathbf{CP}^{N_1 N_2 \cdots N_m - 1}$. Now, let us consider the composite quantum system $\mathcal{Q}^{p}_{m}(N_1, N_2, \ldots, N_m)$ and let the coefficients of $|\Psi\rangle$, namely $\alpha_{i_1,i_2,\ldots,i_m}$, make an array as follows

$$\mathcal{A} = (\alpha_{i_1,i_2,\ldots,i_m})_{1 \leq i_j \leq N_j}, \tag{15}$$

for all $j = 1, 2, \ldots, m$. $\mathcal{A}$ can be realized as the following set $\{(i_1, i_2, \ldots, i_m) : 1 \leq i_j \leq N_j, \forall\, j\}$, in which each point $(i_1, i_2, \ldots, i_m)$ is assigned the value $\alpha_{i_1,i_2,\ldots,i_m}$. Then $\mathcal{A}$ and it's realization is called an m-dimensional box-shape matrix of size $N_1 \times N_2 \times \cdots \times N_m$, where we associate to each such matrix a subring $\mathrm{S}_{\mathcal{A}} = \mathbf{C}[\mathcal{A}] \subset \mathrm{S}$, where S is a commutative ring over the complex number field. For each $j = 1, 2, \ldots, m$, a two-by-two minor about the j-th coordinate of $\mathcal{A}$ is given by

$$\begin{aligned}
\mathcal{C}_{k_1,l_1;k_2,l_2;\ldots;k_m,l_m} &= \alpha_{k_1,k_2,\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_m} \\
&\quad - \alpha_{k_1,k_2,\ldots,k_{j-1},l_j,k_{j+1},\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_{j-1},k_j,l_{j+1},\ldots,l_m} \in \mathrm{S}_{\mathcal{A}}.
\end{aligned} \tag{16}$$

Then the ideal $\mathcal{I}^{m}_{\mathcal{A}}$ of $\mathrm{S}_{\mathcal{A}}$ is generated by $\mathcal{C}_{k_1,l_1;k_2,l_2;\ldots;k_m,l_m}$ and describes the separable states in $\mathbf{CP}^{N_1 N_2 \cdots N_m - 1}$. The image of the Segre embedding $\mathrm{Im}(\mathcal{S}_{N_1,N_2,\ldots,N_m})$ which again is an intersection of families of quadric hypersurfaces in $\mathbf{CP}^{N_1 N_2 \cdots N_m - 1}$ is given by

$$\begin{aligned}
\mathrm{Im}(\mathcal{S}_{N_1,N_2,\ldots,N_m}) &= < \mathcal{C}_{k_1,l_1;k_2,l_2;\ldots;k_m,l_m} > \\
&= \mathcal{V}(\mathcal{C}_{k_1,l_1;k_2,l_2;\ldots;k_m,l_m}).
\end{aligned} \tag{17}$$

In our paper [17], we showed that the Segre variety defines the completely separable states of a general multipartite state. Furthermore, based on this subdeterminant, we define an entanglement measure for general pure bipartite and three-partite states which coincide with generalized concurrence. Let us consider a general multi-qubit state $\mathcal{Q}_m^p(2,\ldots,2)$. For this state the Segre variety is given by equation (17) and

$$\begin{aligned}
\mathrm{Im}(\mathcal{S}_{2,\ldots,2}) &= \mathcal{V}(\mathcal{C}_{1,2;1,2;\ldots;1,2}) \\
&= \overbrace{\mathbf{CP}^1 \times \cdots \times \mathbf{CP}^1}^{m \text{ times}} \simeq \mathbf{S}^2 \times \cdots \times \mathbf{S}^2 \\
&\subset \mathbf{S}^{2^m} \xleftarrow{\mathbf{S}^{2^m-1}} \mathbf{S}^{2^{m+1}-1} \xrightarrow{\mathbf{S}^1} \mathbf{S}^{2^{m+1}-1}/U(1) = \mathbf{CP}^{2^m-1} .
\end{aligned} \tag{18}$$

We can parameterized the sphere $\mathbf{S}^{2^{m+1}-1}$ as a $\mathbf{S}^{2^m-1}$ fiber over $\mathbf{S}^{2^m}$, that is $\mathbf{S}^{2^{m+1}-1} \xrightarrow{\mathbf{S}^{2^m-1}} \mathbf{S}^{2^m}$ which are higher order Hopf fibration. Moreover, we have the following commutative diagram

$$\begin{tikzcd}
\mathbf{S}^{2^{m+1}-1} \arrow[r, "id"] \arrow[d, "\mathbf{S}^1"'] & \mathbf{S}^{2^{m+1}-1} \arrow[d, "\mathbf{S}^{2^m-1}"] \\
\mathbf{CP}^{2^m-1} = \mathbf{S}^{2^{m+1}-1}/U(1) \arrow[r, "\mathbf{S}^{2^m-2}"] & \mathbf{S}^{2^m}
\end{tikzcd}$$

Thus we have established relations between Segre variety and higher order Hopf fibration and separable set of a multi-qubit state. As an example, let us look at a pure three-qubit state. For such state we have

$$\begin{aligned}
\mathrm{Im}(\mathcal{S}_{2,2,2}) &= \mathcal{V}(\mathcal{C}_{1,2;1,2;1,2}) \\
&= \langle \alpha_{1,1,1}\alpha_{2,1,2} - \alpha_{1,1,2}\alpha_{2,1,1}, \alpha_{1,1,1}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,1} \\
&\quad , \alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,1,1}, \alpha_{1,1,2}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,2} \\
&\quad , \alpha_{1,1,2}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,1,2}, \alpha_{1,2,1}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,2,1} \\
&\quad , \alpha_{1,1,1}\alpha_{1,2,2} - \alpha_{1,1,2}\alpha_{1,2,1}, \alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,2,1}\alpha_{2,1,2} \\
&\quad , \alpha_{1,1,2}\alpha_{2,2,1} - \alpha_{1,2,2}\alpha_{2,1,1}, , \alpha_{2,1,1}\alpha_{2,2,2} - \alpha_{2,1,2}\alpha_{2,2,1} \\
&\quad , \alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,1,2}\alpha_{2,2,1}, \alpha_{1,2,1}\alpha_{2,1,2} - \alpha_{1,2,2}\alpha_{2,1,1} \rangle \\
&= \mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1 \simeq \mathbf{S}^2 \times \mathbf{S}^2 \times \mathbf{S}^2 \\
&\subset \mathbf{S}^8 \xleftarrow{\mathbf{S}^7} \mathbf{S}^{15} \xrightarrow{\mathbf{S}^1} \mathbf{S}^{15}/U(1) = \mathbf{CP}^7 .
\end{aligned} \tag{19}$$

This is what we have expected to see. Moreover, we have the following commutative diagram

$$\begin{tikzcd}
\mathbf{S}^{15} \arrow[r, "id"] \arrow[d, "\mathbf{S}^1"'] & \mathbf{S}^{15} \arrow[d, "\mathbf{S}^7"] \\
\mathbf{CP}^7 = \mathbf{S}^{15}/U(1) \arrow[r, "\mathbf{S}^6"] & \mathbf{S}^8
\end{tikzcd}$$

where we have the third Hopf fibration $\mathbf{S}^{15} \xrightarrow{\mathbf{S}^7} \mathbf{S}^8$ for three-qubit state which has been discussed in Ref. [7].

8 Conclusion

In this paper, we have discussed a geometric picture of the separable pure two-qubit states based on Segre variety, conifold, and Hopf fibration. We have shown that these varieties and mappings give a unified picture of two-qubit states. Moreover, we have discussed the geometry and topology of pure multi-qubit states based on multi-projective Segre variety and higher-order Hopf fibration. Thus we have established relations between algebraic geometry, algebraic topology and fundamental quantum theory of entanglement. Perhaps, these geometrical and topological visualization puts entanglement in a broader perspective and hopefully gives some hint about how we can solve the problem of quantify entanglement.

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References

[1] E. Schrödinger, Naturwissenschaften **23**, 807-812, 823-828, 844-849 (1935). Translation: Proc. of APS, **124**, 323 (1980).

[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[3] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[4] D. C. Brody and L. P. Hughston, Journal of Geometry and Physics **38**, 19 (2001).

[5] A. Miyake, Phys. Rev. A **67**, 012108 (2003).

[6] R. Mosseri, R. Dandoloff, e-print quant-ph/0108137.

[7] B. A. Bernevig, H. D. Chen, e-print quant-ph/0302081.

[8] H. Li and F. Van Oystaeyen, *A primer of Algebraic geometry*, Marcel Dekker, Inc., New York, 2000.

[9] C. Musili, *Algebraic geometry*, Hindustan book agency, India, 2001.

[10] K. Ueno, *An introduction to algebraic geometry*, American Mathematical Society, Providence, Rhode Island, 1997.

[11] P. Griffiths and J. Harris, *Principle of algebraic geometry*, Wiley and Sons, New York, 1978.

[12] D. Mumford, *Algebraic Geometry I, Complex Projective Varieties*, Springer-Verlag, Berlin, 1976.

[13] S. Albeverio and S. M. Fei, J. Opt. B: Quantum Semiclass. Opt. **3**, 223 (2001).

[14] E. Gerjuoy, Phys. Rev. A **67**, 052308 (2003).

[15] I. R. Klebanov, E. Witten, e-print hep-th/9807080.

[16] D. R. Morrison, M. R. Plesser , e-print hep-th/9810201.

[17] H. Heydari and G. Björk, J. Phys. A: Math. Gen. **38**, 3203-3211 (2005).